On approximation of topological groups by finite algebraic systems

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Abstract

It is known that locally compact groups approximable by finite ones are unimodular, but this condition is not sufficient, for example, the simple Lie groups are not approximable by finite ones as topological groups. In this paper the approximations of locally compact groups by more general finite algebraic systems are investigated. It is proved that the approximation of locally compact groups by finite semigroups is equivalent to approximation by finite groups and thus not all locally compact groups are approximable by finite semigroups. We prove that any locally compact group is approximable by finite left (right) quasigroups but the approximabilty of a locally compact group by finite quasigroups (latin squares) implies its unimodularity. The question if the unimodularity of a locally compact group implies its approximability by finite quasigroups is open. We prove only that the discrete groups are approximable by finite quasigroups.

1 Introduction

The notion of approximation of a topological group by finite ones was introduced by the second author (cf. the monograph [1] and the bibliography there). It was investigated in details for the case of locally compact abelian (LCA) groups in [1] and for the case of discrete groups in [2]. The approximations of LCA groups were used in [3] for a construction of finite dimensional approxiamtions of pseudodifferential operators. The approximations of discrete groups have some interesting applications in the ergodic theory of group actions [2],[4] and in symbolic dynamics [5]. The approximability of any LCA group by finite abelian groups is proved in [1], the approximability of a huge class of nilpotent Lie groups by finite nilpotent groups is proved in [2]. The class of discrete approximable groups is a proper extension of the class of locally residually finite groups; there exist some non-approximable groups: the Baumslag - Solitar groups, finitely presented infinite simple groups and some others [2]. It was proved in [6] that all approximable locally compact groups are unimodular (the left and right Haar measures coincide). This condition is not sufficient - we have mentioned already that there exist non-approximable discrete groups. It was proved in [4] that the simple Lie groups are not approximable by finite groups as topological groups (since these groups are locally residually finite they are approximable as discrete groups).

In this paper we investigate the approximation of locally compact groups by more general universal algebras with one binary operation. We prove that any locally compact group is approximable by finite left (right) quasigroups - the algebras, that have left (right) division[1]. Then we consider approximations of more general topological algebras and prove that if any locally compact left (right) quasigroup A that has the operation of taking left (right) inverse element (satisfying the left (right) cancellation law[2]) is approximable by finite left (right) quasigroups then there exists a positive non-trivial linear functional I on $\mathbf{C}_0(A)$ (the space of all continuous functions on A with compact support) such that $I(f) \geq I(l_h(f))$ $(I(f) \geq I(r_h(f)))$ for any non-negative $f \in C_0(A)$ and any $h \in A$. Here l_h (r_h) is the left (right) shift on $\mathbf{C}_0(A)$, i.e. $l_h(f)(a) = f(h \circ a)$ $(r_h(f)(a) = f(a \circ h)$, where $\circ$ is the operation in A.

These inequalities imply immediately that if A is a group, then I is left (right) invariant and thus if the group A is approximable by finite quasigroups, i.e. algebraic systems that are left and right quasigroups simultaneously, then A is unimodular. So, any locally compact group that is approximable by finite quasigroups is unimodular. It is an interesting open question if the approximability by finite quasigroups implies

*The work of the second author was supported in part by NSF Grant DMS-9970009

[1]See Definition 2(1)

[2]See Definition 2(4)

the unimodularity of a locally compact group. We can only say that the class of locally compact groups that are approximable by finite quasigroups is larger than the class of locally compact groups approximable by finite groups. We prove that all discrete groups are approximable by finite quasigroups.

We prove also that the approximabilty of a locally compact group by finite semigroups implies its approximability by finite groups.

In the proofs of mentioned results we use the language of nonstandard analysis that allows to simplify essentially these proofs. The necessary notions and results of nonstandard analysis can be found in the monographs [7] or [8]

The results of this paper were discussed at the seminar "Nonstandard analysis" in the UIUC. The authors are grateful to Prof. C.W.Henson and Prof. P.Loeb for many important remarks.

2 Formulation of the main results

By an algebra we mean here a universal algebra that contains only one binary operation.

Let $< A, \circ >$ be a locally compact Hausdorff topological universal algebra, $C \subset A$ - a compact subset, $\mathcal{U}$ - a finite cover of C by open sets, $< H, \odot >$ - a finite universal algebra. In what follows we will omit the symbols of operation and denote these algebras by A and H respectively.

Definition 1 1. *A set $M \subset A$ is called a $(C,\mathcal{U})$-grid iff*

$$\forall U \in \mathcal{U}\, ((C \cap U \neq \emptyset) \Rightarrow \exists m \in M\ (m \in U)).$$

2. *A map $j : H \to A$ is called a $(C,\mathcal{U})$-homomorphism iff*

$$\forall x, y \in H\, ((j(x), j(y), j(x) \circ j(y) \in C) \Rightarrow \exists U \in \mathcal{U}\ \ (j(x \odot y) \in U \wedge j(x) \circ j(y) \in U)).$$

3. *We say that the pair $< H, j >$ is a $(C,\mathcal{U})$-approximation of G if $j(H)$ is a $(C,\mathcal{U})$-grid and $j : H \to G$ is a $(C,\mathcal{U})$-homomorphism.*

4. *Let $\mathcal{K}$ be a class of finite algebras. We say that A is approximable by the systems of the class $\mathcal{K}$ if for any compact $C \subset A$ and for any finite cover $\mathcal{U}$ of C by open sets there exists a $(C,\mathcal{U})$-approximation $< H, j >$ of A such that $H \in \mathcal{K}$ and j is an injection.*

Remark 1 *It is easy to see that a similar definition can be formulated for any topological universal algebra and it is not necessary to assume that approximated algebras are finite. For example, the approximations of discrete groups by amenable ones were introduced in [4]. The approximations of universal algebras with finite signatures will be considered in another paper.*

It is easy to see that the following propositions hold.

Proposition 1 *If A is separable as a topological space then A is approximable by algebras of a class $\mathcal{K}$ iff there exist a sequence of finite algebras $< H_n, \circ_n >$, $H_n \in \mathcal{K}$ and a sequence of injections $j_n : H_n \to A$ such that for any compact $C \subset A$ and for any finite cover $\mathcal{U}$ of C by open sets there exists an $n_0 \in \mathbf{N}$, such that for any $n > n_0$ $< H_n, j_n >$ is a $(C,\mathcal{U})$-approximation of A.*

Proposition 2 *A discrete algebra A is approximable by algebras of a class $\mathcal{K}$ iff for any finite subset $S \subset A$ there exist an algebra $H \in \mathcal{K}$ and an injection $j : S \to H$ such that*

$$\forall s_1, s_2 \in S\, \big(s_1 \circ s_2 \in S \Rightarrow j(s_1 \circ s_2) = j(s_1) \odot j_(s_2)\big)$$

The definition of approximation of a LC group by finite algebras can be simplified a little. Let G be a LC group. We will denote by $\cdot$ the multiplication in G and use the usual notations

$$XY = \{x \cdot y \mid x \in X,\, y \in Y\}$$

$$X^{-1} = \{x^{-1} \mid x \in X\}$$

$$gX = \{g \cdot x \mid x \in X\}$$

for $X, Y \subset G,\ g \in G$.

Definition 2 *Let $C \subset G$ be a compact, U - a relatively compact neighborhood of the unit in G, and H - a finite algebra.*

1. *We say that a set $M \subset G$ is an U-grid of C iff $C \subset MU$.*
2. *A map $j : H \to G$ is called a (C, U)-homomorphism if*

$$\forall x, y \in H \ \ ((j(x), j(y), j(x) \cdot j(y) \in C) \Rightarrow (j(x \odot y) \in j(x)j(y)U))$$

3. *We say that the pair $< H, j >$ is a (C, U)-approximation of G if $j(H)$ is an U-grid of C and $j : H \to G$ is a (C, U)-homomorphism.*
4. *Let $\mathcal{K}$ be a class of finite algebras. We say that G is approximable by the systems of the class $\mathcal{K}$ if for any compact $C \subset G$ and for any neighborhood of the unit U there exists a (C,U)-approximation $< H, j >$ of G such that $H \in \mathcal{K}$ and j is an injection.*

Proposition 3 *A locally compact group G is approximable by the systems of a class $\mathcal{K}$ in the sense of Definition 2 iff it is approximable by the systems of $\mathcal{K}$ in the sense of Definition 1.*

This proposition will be proved in the section 4.

Definition 3 1. *We say that algebra A is an l-quasigroup (r-quasigroup) iff for all $a, b \in A$ the equation $a \circ x = b$ ($x \circ a = b$) has the unique solution $x = /(b, a)$ ($x = \backslash(b, a)$).*

2. *If the functions $\circ(\cdot, \cdot) : A^2 \to A$ $/(\cdot, \cdot) : A^2 \to A$, $\backslash(\cdot, \cdot) : A^2 \to A$) are continuous then we say that A is a topological l-quasigroup (r-quasigroup).*
3. *We say that $(A, \circ)$ is a (topological) quasigroup iff it is a (topological) l- and r-quasigroup simultaneously.*
4. *We say that l-quasigroup (r-quasigroup) A satisfies the l-cancellation law (r-cancellation law) iff there exists a function $b : A \to A$ such that*

$$\forall a \forall x \ b(a) \circ (a \circ x) = x \quad (\forall a \forall x \ (x \circ a) \circ b(a) = x).$$

We write b_a instead of $b(a)$.

There is a huge literature concerning quasigroups, cf., for example, [9]. The operation table of a finite quasigroup is a latin square - an $n \times n$-table of n elements $\{a_1, \ldots, a_n\}$ such that all elements in each row and in each column are distinct. An $n \times n$-table with this property that contains more than n elements is called a latin subsquare. It is known [10] that any $n \times n$ latin subsquare with k distinct elements can be completed to an $r \times r$ latin square, where $r = \max\{2n, k\}$. This fact together with Proposition 2 implies immediately the following proposition.

Proposition 4 *Any discrete quasigroup, and thus any discrete group, is approximable by finite quasigroups.*

Theorem 1 *Any locally compact group G is approximable by finite l-quasigroups (r-quasigroups).*

This theorem will be proved in the section 3.

Theorem 2 *Let A be a locally compact l-quasigroup (r-quasigroup) that satisfies the l-cancellation (r-cancellation) law and that is approximable by finite l-quasigroups (r-quasigroups). Then there exists a positive bounded non-trivial linear functional I on $\mathbf{C}_0(A)$ that satisfies the inequality*

$$I(f) \geq I(l_h(f)) \quad (I(f) \geq I(r_h(f))) \tag{1}$$

for any non-negative $f \in C_0(A)$ and any $h \in A$.

If A is a locally compact quasigroup that satisfies the both cancellation laws and that is approximable by finite quasigroups then I satisfies the both inequalities (1) simultaneously.

This theorem will be proved in the section 5.

Proposition 5 *If A is a group and I is a functional on $\mathbf{C}_0(A)$ that satisfies (1) then I is left (right) invariant.*

Proof

$$I(f) \geq I(l_h(f)) \geq I\left(l_{h^{-1}}\left(l_h(f)\right)\right) = I(f) \quad \square$$

Remark 2 *Theorems 1, 2 and Proposition 5 imply the existence of the Haar measure on a locally compact group. Indeed, the proof of Theorem 1 given in the section 4 includes some ideas and constructions of the proof of existence of the Haar measure that is contained in the famous monograph [11]. One more proof of existence of the Haar measure base on nonstandard analysis is contained in [12].*

Corollary 1 *Any locally compact group G approximable by finite quasigroups is unimodular.*

This corollary generalizes the result of [6] that any locally compact group approximable by finite groups is unimodular. Proposition 4 shows that the class of groups approximable by finite quasigroups is larger then the class of groups approximable by finite groups since there exist discrete groups that are not approximabale by finite groups [2]

Conjecture A locally compact group is unimodular iff it is approximable by finite quasigroups.

Theorem 3 *A locally compact group is approximable by finite semigroups iff it is approximable by finite groups.*

This theorem will be proved in the section 6.

3 Proof of Theorem 1

In this section G is a locally compact group.

To prove Theorem 1 we introduce the following construction. Given a neighborhood of the unit U and a compact C we find a finite U- grid of C (Definition 2 (1)) $F \subset G$ and a collection $\{A_{g,h} \subset F\ ;\ g,h \in F\}$, such that the following lemma holds.

Lemma 1 *For any neighborhood of the unity U and any compact C there exist a finite $F \subset G$, and a collection $\{A_{g,h} \subset F\ ;\ g,h \in F\}$, satisfying the following conditions:*

1. *F is an U-grid of C;*
2. *if $g,h \in C \cap F$, then $A_{g,h} \subset ghU$;*
3. *$\forall g \in F \forall S \subset F\ \ |\bigcup_{h\in S} A_{g,h}| \geq |S|$.*

We also need the well known combinatorial Theorem of P. Hall (the Marriage Lemma), see, for example, [10].

Definition 4 *Let $F_i \subset F,\ i = 1,\ldots,m$. We say that the sequence $F_1, F_2, ..., F_m$ has a system of distinct representatives (SDR) iff we can chose m-permutation of F $a_1, a_2, ..., a_m$, such that $a_i \in F_i$ for $i = 1, ..., m$. (Definition of m-permutation requires that $a_i \neq a_j$ for $i \neq j$.)*

Theorem 4 *The subsets $F_1, F_2, ...F_m$ have an SDR if and only if for each $S \subseteq \{1, 2, ..., m\}$ the following inequality holds*

$$|\bigcup_{k\in S} F_k| \geq |S|$$

Remark 1 *Nonstandard analysis versions of P.Hall's theorem were investigated in [13].*

Lemma 1 (3) and Theorem 4 imply that the set F may be equipped with operation $\odot$, satisfying the definition of l-quasigroup. Indeed by the condition 3) the system $\{A_{g,h} \mid h \in F\}$ satisfies Theorem 4 for any fixed $g \in F$ and thus for any $g, h \in F$ there exists $g \odot h \in A_{g,h}$ such that for any $g \in F$ $\{g \odot h \mid h \in F\}$ is a permutation of F. Thus $< F, \odot >$ is an l-quasigroup. The conditions 1) and 2) of Lemma 1 imply the l-quasigroup $< F, \odot >$ with the identical inclusion is a (C, U)-approximation of G, see Definition 2(3).

So, to complete the proof of Theorem 1 we have only to prove the lemma 1.

Let $O \subset G$ be a neighborhood of the unity and $A \subset G$. Denote $(A : O)$ the minimal number n, such that there exist F, $|F| = n$, $A \subseteq FO$.

In the following Propositions 6 and 7 we assume that

- O is a neighborhood of the unity;
- K is a compact;
- F is a finite set, $|F| = (K : 0)$ and $K \subset FO$ (F is an optimal O-grid of K).

Proposition 6 *Let $S \subset F$, then $(SO : O) = |S|$.*

Proof. It is clear that $(SO : O) \leq |S|$. Suppose, that $(SO : O) < |S|$, then

$$K \subset SO \cup \bigcup_{x \in F \setminus S} xO,$$

and we can cover K with less then $|F|$ elements. □

Proposition 7 *Let $M \subset K$. Then $|MO^{-1} \cap F| \geq (M : O)$.*

Proof One has $M \subseteq K \subseteq FO$. It means that $\forall x \in M \; \exists f \in F \exists \; \epsilon \in O \;\; x = f\epsilon$, or $f = x\epsilon^{-1}$ so, $f \in MO^{-1}$. Consequently, $MO^{-1} \cap F$ is an O-grid of M. So, $(M : O) \leq |MO^{-1} \cap F|$. □

Proof of Lemma 1. Given a neighborhood of the unity $U \subset G$ and a compact $C \subset G$ one can chose a neighborhood of the unity O and a compact K, such that

- $OO^{-1} \subset U$;
- $C^2 \subset K$;
- $CU \subset K$.

Let F be an optimal O-grid of K. Define the sets $A_{g,h}$ as follows:

$$A_{g,h} = \begin{cases} ghOO^{-1} \cap F, & \text{if } g, h \in C \\ F, & \text{otherwise.} \end{cases}$$

It is easy to see that F is U-grid of C and item 2) of Lemma 1 is also satisfied.

The proof of item 3). Nontrivial case is when $g \in C$ and $S \subset C$. By Proposition 6 $(SO : O) = |S|$, consequently, $(gSO : O) = |S|$. Then, by Proposition 7,

$$|S| \leq |gSOO^{-1} \cap F| = |\bigcup_{h \in S} A_{g,h}|. \;\; \square$$

4 Nonstandard analysis approach to approximation of algebras

In this section we introduce a brief exposition of nonstandard analysis (see, for example, [1],[7] or [8] for details).

Let λ be an infinite cardinal. Consider a λ^+-saturated nonstandard extension ${}^*\mathcal{V}$ of the standard universe $\mathcal{V}$. Recall that ${}^*\mathcal{V}$ is λ^+-saturated if for any family $\mathcal{X}$ of *internal* sets (i.e. the sets that are elements of ${}^*\mathcal{V}$), such that $|\mathcal{X}| \leq \lambda$ and $\mathcal{X}$ has the finite intersection property, then it follows that $\bigcap_{X \in \mathcal{X}} X \neq \emptyset$.

There exists an embedding $^*: \mathcal{V} \to {}^*\mathcal{V}$ that satisfies the *transfer principle*. The image of an element $v \in \mathcal{V}$ under this embedding is denoted by ${}^*v$ and is called the nonstandard extension of v. A proposition $\varphi(v_1, \ldots, v_n)$ is called *internal* if it is a statement about $v_1, \ldots, v_n$ that is formulated in usual mathematical terms - not including such notions as "standard element", "nonstandard extension", etc. More formally φ can be a formula of the language of the set theory, or of the language of the theory of superstructures, or of the language of the elementary analysis [1], etc., depending on what kind of standard universe $\mathcal{V}$ we consider.

Transfer principle If φ is an internal formula and $v_1, \ldots, v_n \in \mathcal{V}$ then $\varphi(v_1, \ldots, v_n)$ holds in $\mathcal{V}$ iff $\varphi({}^*v_1, \ldots, {}^*v_n)$ holds in ${}^*\mathcal{V}$.

Example Let $n \in \mathbf{N}$ and $B_n = \{\xi \in \mathbf{R} \mid \xi > n\}$. By the transfer principle, ${}^*B_n = \{\xi \in {}^*\mathbf{R} \mid \xi > {}^*n\}$ and ${}^*n \in {}^*\mathbf{N}$. The elements of ${}^*\mathbf{N}$ are called the hypernatural numbers and the elements of ${}^*\mathbf{R}$ - the hyperreals. Usually the notation ${}^*B$ is used only for the nonstandard extension of a standard set B. The nonstandard extension of a standard element b is denoted by b also. So, if $\xi \in B_n$ then by the transfer principle $\xi \in {}^*B_n$ and thus $B_n \subset {}^*B_n$.

The countable sequence $\{{}^*B_n \mid n \in \mathbf{N}\}$ has obviously the finite intersection property and thus by saturation $\mathrm{M} = \bigcap\limits_n {}^*B_n \neq \emptyset$. Let $-\mathrm{M} = \{-x \ : \ x \in \mathrm{M}\}$. The elements of $\mathrm{M} \cup -\mathrm{M}$ are called the *infinite* elements of ${}^*\mathbf{R}$ since if $\eta \in \mathrm{M}$ then

$$\forall \xi \in \mathbf{R}\ (|\eta| > |\xi|) \tag{2}$$

The elements of ${}^*\mathbf{R}$ inverse to infinite elements (and ${}^*\mathbf{R}$ is an ordered field by Transfer principle) are called the *infinitesimals*. It follows from (2) that if α is an infinitesimal then

$$\forall \xi \in \mathbf{R}\ (\xi > 0 \Rightarrow |\alpha| < \xi).$$

The set of all infinitesimals is called *the monad of zero* and denoted by $\mu(0)$. The sets M and $\mu(0)$ are not internal - such sets are called *external*. Indeed if $\mu(0)$ would be internal, then, being bounded from above, it must have the supremum by the transfer principle. But it is easy to see that both conjectures: 1) $\sup \mu(0)$ is an infinitesimal and 2) $\sup \mu(0)$ is not infinitesimal - lead to a contradiction. Consider one more construction, similar of which will be used in the proof of Theorem 5. Let $\mathcal{B}_n = \{B_m \ : \ m \geq n\}$. By saturation $\bigcap\limits_n {}^*\mathcal{B}_n \neq \emptyset$. One can verify that every $X \in \bigcap\limits_n {}^*\mathcal{B}_n$ is an internal set of the form $X = {}^*B_\nu$ for some $\nu \in {}^*\mathbf{N} \setminus \mathbf{N}$.

Two elements $\xi, \eta \in {}^*\mathbf{R}$ are *infinitely close* ($\xi \approx \eta$) if $\xi - \eta \in \mu(0)$. In particular, $\xi \in \mu(0)$ iff $\xi \approx 0$.

The elements of ${}^*\mathbf{R}$ that are not infinite are called *bounded* or *finite*. It can be proved that any bounded element ξ is infinitely close to the unique standard element α. This α is called *the standard part* of ξ or *the shadow* of ξ and denoted by ${}^\circ\xi$. Now it is easy to prove that any hypernatural number that is not infinite is standard.

An internal set B is called *hyperfinite* if there exists an $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ and an internal bijection φ : $\{0, 1, \ldots, n-1\} \to B$. Then we say that n is the cardinality of B and write $|B| = n$. When we deal with internal sets we use the term "cardinality" for internal cardinality. By the transfer principle the hyperfinite sets have many features of standard finite sets. For example, an internal subset of a hyperfinite set is hyperfinite itself. Any hyperfinite set B of reals has the maximal and the minimal elements and the sum $\sum\limits_{b \in B} b$ is defined. If $B = B_1 \cup B_2$, where B_1 and B_2 are internal sets and $B_1 \cap B_2 = \emptyset$ then

$$\sum_{b \in B} b = \sum_{b \in B_1} b + \sum_{b \in B_2} b.$$

We consider now the nonstandard extension ${}^*X$ of a standard topological space X of the weight (the cardinality of a minimal base of topology on X) less or equal to λ. If $x \in X$ is not an isolated point of X then for any open $U \ni x$ the set $U \setminus \{x\} \neq \emptyset$. Let

$$\mu(x) = \bigcap \{\, {}^*U \mid U \ni x,\ U \text{ is open}\}.$$

The set $\mu(x)$ (external, if x is not an isolated point) is called *the monad* of x. The λ^+-saturation of the nonstandard universe ${}^*\mathcal{V}$ implies immediately that $\mu(x)$ contains some nonstandard elements of ${}^*X$ if x is not an isolated point. We say that an element $y \in {}^*X$ is nearstandard if $y \in \mu(x)$ for some standard $x \in X$. The set of all nearstandard elements of ${}^*X$ is denoted by $\mathrm{ns}({}^*X)$. Obviously $\mathrm{ns}({}^*X) = \bigcup_{x \in X} \mu(x)$. If $y \in \mu(x)$ then we say as before that x is the standard part or the shadow of y and write $x = {}^\circ y$. We say that two nearstandard elements y_1 and y_2 are infinitely close ($y_1 \approx y_2$) if ${}^\circ y_1 = {}^\circ y_2$.

We list the well known properties ([1],[7],[8]) of ${}^*X$ in the following

Proposition 8 *The following properties hold.*

1. *A set $U \subset X$ is open iff $\forall x \in U\,(\mu(x) \subset {}^*U)$.*
2. *A set $F \subset X$ is closed iff $\forall x \in X\,(\mu(x) \cap {}^*F \neq \emptyset \Rightarrow x \in F)$.*
3. *X is compact iff $ns({}^*X) = {}^*X$.*
4. *X is locally compact iff $ns({}^*X) = \bigcup\{{}^*C \mid C \subset X,\ C \text{ is compact}\}$.*
5. *If Y is a topological space of the weight less or equal to λ then a mapping $f : X \to Y$ is continuous if $\forall x \in X, \xi \in {}^*X\,(\xi \approx x \Longrightarrow {}^*f(\xi) \approx f(x))$.*

Let us return now to a locally compact algebra A. Assume that the cardinality of its topology (the family of all open sets) is λ and consider the nonstandard extension ${}^*A$ of A in our λ^+-saturated nonstandard universe ${}^*\mathcal{V}$.

Theorem 5 *The algebra A is approximable by finite systems of a class $\mathcal{K}$ iff there exists a hyperfinitie system $H \in {}^*\mathcal{K}$ and an internal injection $j : H \to A$ that satisfy the following properties.*

1. *$\forall a \in A \exists h \in H\ (j(h) \approx a)$.*
2. *$\forall h_1, h_2 \in j^{-1}(ns({}^*A))\,(j(h_1 \odot h_2) \approx j(h_1) \circ j(h_2))$*

Proof $\Rightarrow$ Assume that A is approximable by finite systems of a class $\mathcal{K}$. Let $\mathcal{H}$ be the set of all pairs $(C,\mathcal{U})$, where $C \subset A$ is a compact and $\mathcal{U}$ - a finite cover of C by open sets. Notice that the cardinality of $\mathcal{H}$ is λ. Consider the following preordering $\leq$ of $\mathcal{H}$:

$$(C,\mathcal{U}) \leq (C',\mathcal{U}') \Leftrightarrow (C \supset C') \wedge \forall U \in \mathcal{U}\,(U \cap C' \neq \emptyset \Rightarrow \exists V \in \mathcal{U}'(U \subset V))\,. \tag{3}$$

Let us show that

$$\forall (C_1\mathcal{U}_1),\ (C_2,\mathcal{U}_2) \exists (C,\mathcal{U})\,((C,\mathcal{U}) \leq (C_1,\mathcal{U}_1) \wedge (C,\mathcal{U}) \leq (C_2,\mathcal{U}_2))\,. \tag{4}$$

Let $C = C_1 \cup C_2$ and $\mathcal{U} = \{U \setminus C_1 \mid U \in \mathcal{U}_2\} \cup \{U \setminus C_2 \mid U \in \mathcal{U}_1\} \cup \{U \cap V \mid U \in \mathcal{U}_1, V \in \mathcal{U}_2, U \cap V \neq \emptyset\}$.

It is enough to show that $(C,\mathcal{U}) \leq (C_1,\mathcal{U}_1)$. By the construction $C \supset C_1$ and if $W \in \mathcal{U}$ and $W \cap C_1 \neq \emptyset$ then W is either of the form $U \setminus C_2,\ U \in \mathcal{U}_1$ or of the form $U \cap V,\ U \in \mathcal{U}_1, V \in \mathcal{U}_2$. In both cases (3) holds.

Let $\mathcal{H}(C,\mathcal{U}) = \{(C',\mathcal{U}') \mid (C',\mathcal{U}') \leq (C,\mathcal{U})\}$. The family $\{\mathcal{H}(C,\mathcal{U}) \mid (C,\mathcal{U}) \in \mathcal{H}\}$ is of cardinality λ and has the finite intersection property by (4). By λ^+-saturation

$$\bigcap_{(C,\mathcal{U}) \in \mathcal{H}} {}^*\mathcal{H}(C,\mathcal{U}) \neq \emptyset$$

and thus there exists a pair $(C_0,\mathcal{U}_0) \in {}^*\mathcal{H}$ such that $\forall (C,\mathcal{U}) \in \mathcal{H}\ (C_0,\mathcal{U}_0) \leq ({}^*C,\ {}^*\mathcal{U})$. By the definition of $\leq$ the compact $C_0 \supset \bigcup\{{}^*C \mid C \subset A,\ C \text{ is compact}\} = \mathrm{ns}({}^*A)$ by Proposition 8(4).

Let us show that

$$\forall a \in A \forall U \in \mathcal{U}_0 (a \in U \Rightarrow U \subset \mu(a)). \tag{5}$$

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Indeed, let V be any standard neighborhood of a. Consider $C = \{a\},\ \mathcal{U} = \{V\}$. Since $(C_0,\mathcal{U}_0) \leq ({}^*C,\ {}^*\mathcal{U})$, if $a \in U \in \mathcal{U}_0$ then by (3) $U \subset {}^*V$. Thus $U \subset \bigcap\{{}^*V \mid V \ni a, \mathcal{V} \text{ is open}\} = \mu(a)$.

By the transfer principle there exist a hyperfinite algebra H and an internal injection $j : H \to {}^*A$ such that $< H, j >$ is a $(C_0, \mathcal{U}_0)$-approximation of ${}^*A$. We have to show that $< H, j >$ satisfies the properties 1) and 2) of the theorem.

By Definition 1 and the transfer principle since $a \in C_0$, $\exists U \in \mathcal{U}_0 \exists h \in H(a, j(h) \in U)$. By (5) $a \approx j(h)$ and thus the property 1) holds.

Let $j(h_1), j(h_2) \in \mathrm{ns}({}^*A)$. Since $j(h_1), j(h_2), j(h_1) \circ j(h_2) \in C_0$ (due to the continuity of $\circ$) we have

$$\exists U \in \mathcal{U}_0 \ \ (j(h_1 \odot h_2), j(h_1) \circ j(h_2) \in U)\,.$$

Let us show that $U \subset \mu(a)$ for some $a \in A$. This will prove the property 2). Put $a = {}^\circ(j(h_1) \circ j(h_2))$. It is enough to show that for an arbitrary open set $V \ni a$ holds $U \subset {}^*V$. Consider an open relatively compact $W \ni a$ such that $\overline{W} \subset V$, where $\overline{W}$ is the closure of W. Such W exists because a locally compact space is regular. Notice that $j(h_1) \circ j(h_2) \in {}^*W$ since $j(h_1) \circ j(h_2) \approx a$ by Proposition 8. Let $C = \overline{W}$ and $\mathcal{U} = \{V\}$. We have $(C_0, \mathcal{U}_0) \leq ({}^*C, {}^*\mathcal{U})$, $U \ni j(h_1) \circ j(h_2)$ and thus $U \cap {}^*C \neq \emptyset$. Now by (3) and the transfer principle $U \subset {}^*V$.

$\Leftarrow$ Let $< H, j >$ satisfy the properties 1) and 2) of the theorem, $C \subset A$ be a compact and $\mathcal{U}$ - a finite cover of C by open sets. We have to show that $j(H)$ is a $({}^*C, {}^*\mathcal{U})$-grid and $j : H \to {}^*A$ is a $({}^*C, {}^*\mathcal{U})$-homomorphism. Then the theorem will be proved by the transfer principle (working now the opposite direction of the first part of the proof).

Let ${}^*C \cap {}^*U \neq \emptyset$ then, by the transfer principle $\exists c \in C \cap U$. By the property 1) $\exists h \in H\,(j(h) \approx c)$. Thus $j(h) \in {}^*U$ by Proposition 8. This proves that $j(H)$ is a $({}^*C, {}^*\mathcal{U})$-grid.

Let $j(h_1), j(h_2), j(h_1) \circ j(h_2) \in {}^*C$ then by Proposition 8(3) $a_1 = {}^\circ j(h_1)$, $a_2 = {}^\circ j(h_2)$, $a = {}^\circ j(h_1) \circ j(h_2) \in C$. By the property 2) $j(h_1 \odot h_2) \approx j(h_1) \circ j(h_2) \approx a$. Thus if $a \in U \in \mathcal{U}$ then $j(h_1 \odot h_2), j(h_1) \circ j(h_2) \in {}^*U$ and thus $< H, j >$ is a $({}^*C, {}^*\mathcal{U})$-homomorphism. $\Box$

Proposition 9 *A locally compact group G is approximable by finite algebras of a class $\mathcal{K}$ in the sense of Definition 2 iff there exist a hyperfinitie system $H \in {}^*\mathcal{K}$ and an internal injection $j : H \to A$ that satisfy the properties 1) and 2) of Theorem 5.*

Proposition 3 follows immediately from Theorem 5 and Proposition 9.

Proof. The proof of Proposition 9 is very similar to the proof of Theorem 5 but simpler, so we will sketch briefly the main points.

$\Rightarrow$ By the λ-saturation of the nonstandard universe there exists a compact set $C \subset {}^*G$ such that $\mathrm{ns}({}^*G) \subset C$ and an open neighborhood $U \subset {}^*G$ of the unit e such that $U \subset \mu(e)$. By the transfer principle there exists a hyperfinite (C, U)-approximation $< H, j >$ of ${}^*G$ such that $H \in {}^*K$. It is easy to see that $< H, j >$ satisfies the properties 1) and 2) of Theorem 5.

$\Leftarrow$ Let $< H, j >$ be a hyperfinite algebra that satisfies the properties 1) and 2) of Theorem 5 and $H \in {}^*K$. Then it is easy to see that for any standard $C \subset G$ and any neighborhood $U \subset G$ of the unit $< H, j >$ is a $({}^*C, {}^*U)$-approximation of ${}^*K$. Now by the transfer principle the condition 4) of Definition 2 holds $\Box$

5 Proof of Theorem 2

We start with construction of a functional I that satisfies the condition of Theorem 2. We will consider only the case of a locally compact l-quasigroup A, approximable by finite l-quasigroups. The case of r-quasigroups is similar.

Let $< H, j >$ be a hyperfinite l-quasigroup that satisfies Theorem 5. Let $V \subset A$ be a compact set with a nonempty interior i.e. there exists a nonempty open set $U \subset V$ and thus, by the regularity of the topological space A there exists an open W such that $\overline{W} \subset U$.

We will write $W \sqsubset D$ if $\overline{W}$ is a subset of the interior of D. In the proofs we will often use the following
Statement. If $W \sqsubset D$, $x \in {}^*W$ and $y \approx x$ then $y \in {}^*D$.
The statement easily follows from Proposition 8.

Let $\Delta^{-1} = |j^{-1}({}^*V)|$. Define the functional $I(f)$ as follows:

$$I(f) = {}^\circ\left(\Delta \sum_{h \in H} {}^*f(j(h))\right). \tag{6}$$

The proof of Theorem 2 follows from the following two lemmas: Lemma 2 and Lemma 3.

Lemma 2 *The functional $I(\cdot)$ is a Radon measure on $C_0(A)$.*

We need three following propositions.

Proposition 10 *Let $D \subseteq A$ be compact and $U \sqsubset D$ be an open set. Then for all $a \in A$ the following inequality holds*
$$|j^{-1}(a \circ {}^*U)| \leq |j^{-1}({}^*D)|.$$

Proof. Let $x \in j^{-1}(a \circ {}^*U)$, or $j(x) \in a \circ {}^*U \subset \mathrm{ns}({}^*A)$. By the left cancellation law and the transfer principle $b_a \circ j(x) \in {}^*U$, where $b_a \in A$ does not depend on x. By Theorem 5 there exists $\beta \in H$, such that $b_a \approx j(\beta)$. So, $b_a \circ j(x) \approx j(\beta \circ x) \in {}^*D$ because $U \sqsubset D$. Consequently, $\beta \circ (j^{-1}(a \circ {}^*U)) \subset j^{-1}({}^*D)$, but the function $l_\beta(x) = \beta \circ x$ is an injection, because H is an l-quasigroup. □.

Proposition 11 *Let $X, Y \subset A$ be compact sets and Y has the nonempty interior. Then there exists $0 < C_{X,Y} \in \mathbf{R}$, such that*
$$\frac{|j^{-1}({}^*X)|}{|j^{-1}({}^*Y)|} \leq C_{X,Y}.$$

Proof. Take an open $U \sqsubset Y$. By the definition of a topological l-quasigroup satisfying the left cancellation law the mapping $l_a : A \to A$ is a continuous homeomorphism for any $a \in A$. Moreover since l_{b_a} is the inverse mapping to l_a. Thus $l_a(U) = a \circ U$ is an open set for any $a \in A$. By the definition of an l-quasigroup $\forall z \in A\ A \circ z = A$. Thus $A \circ U$ covers X and so, there exists a finite set $F \subset A$ such that $X \subset F \circ U$. It means that ${}^*X \subset F \circ {}^*U$ (${}^*F = F$). Consequently,
$$|j^{-1}({}^*X)| \leq \sum_{\alpha \in F} |j^{-1}(\alpha \circ {}^*U)|,$$
and, by Proposition 10 $|j^{-1}({}^*X)| \leq |F| \cdot |j^{-1}({}^*Y)|$. So, one can take $C_{X,Y} = |F|$. □

Proposition 12 *Let $\phi : H \to {}^*\mathbf{R}$ satisfy the following conditions:*

1. *$\forall h \in H\ \varphi(h) \geq 0$;*
2. *$j(supp(\varphi)) \subset {}^*S$, where $S \subset A$ is a compact;*
3. *If $j(h) \in {}^*D$, then $\varphi(h) > \alpha$ for some compact $D \subset A$ with the nonempty interior and some $\alpha \in {}^*\mathbf{R}$.*

Then
$$\frac{1}{C_{V,D}} \alpha \leq \Delta \sum_{h \in H} \phi(h) \leq C_{S,V} \sup(\phi). \tag{7}$$

The proof of Lemma 2 follows immediately from Proposition 12.

Indeed, take $\varphi(h) = {}^*f(j(h))$ for any $0 < f \in \mathbf{C}_0(A)$. Then φ satisfies the conditions of Proposition 12. Obviously $\varphi(h) \geq 0$, $S = \mathrm{supp}(f)$. Since $f > 0$ there exists a point $a \in A$ such that $f(a) > 0$ and thus there exist an open $U \ni a$ and a positive α such that $\forall b \in U\ f(b) > \alpha$. Take any relatively compact open W such that $\overline{W} \subset U$. Then $D = \overline{W}$ satisfies the condition 3 of Proposition 12.

By (6) and the first inequality (7) $I(f) \neq 0$. By the second inequality (7) the linear functional I is continuous.

Proof of Proposition 12. Recall that $\Delta^{-1} = |j^{-1}({}^*V)|$.By Proposition 11:
$$\Delta \sum_{h \in H} \phi(h) \geq \Delta \sum_{j(h) \in {}^*D} \phi(h) \geq \frac{\alpha}{C_{V,D}}.$$

This proves the first inequality (7). The second inequality (7) is obtained as follows:
$$\Delta \sum_{h \in H} \phi(h) = \Delta \sum_{j(h) \in {}^*S} \phi(h) \leq C_{S,V} \cdot \sup_x \phi(x). \quad \square$$

Lemma 3 *The functional I, defined by (6), satisfies the inequality $I(f) \geq I(l_a(f))$ for any non-negative $f \in C_0(A)$.*

Proof. For $X \subset A$, $z \in A$ let us denote $/(X,z) = \{/(x,z) \ : \ x \in X\}$, see Definition 3. Let $S \subset A$ be a compact, such that there exists an open set $U \subset A$ with the property $/(\mathrm{supp}(f), a) \subset U \sqsubset S$. Let $h \in H$ such that $j(h) \approx a \in A$. First of all the following equality holds:

$$^\circ\left(\Delta \sum_{x \in H} {}^*f(a \circ j(x)) - \Delta \sum_{x \in H} {}^*f(j(h) \circ j(x))\right) = 0. \tag{8}$$

To prove it let $\varphi(x) = |{}^*f(a \circ j(x)) - {}^*f(j(h) \circ j(x))|$ and apply Proposition 12 as follows. By the continuity of $\circ$ and $/(\cdot,\cdot)$ in A, we have

$a \circ j(x) \in \mathrm{ns} \Leftrightarrow j(x) \in \mathrm{ns} \Leftrightarrow j(h) \circ j(x) \in \mathrm{ns}$. Let us show now that $j(\mathrm{supp}(\varphi)) \subset {}^*S$.

It is enough to show that

$$j(x) \notin {}^*S \Longrightarrow {}^*f(a \circ j(x)) = {}^*f(j(h) \circ j(x)) = 0. \tag{9}$$

Assume that $a \circ j(x) \in *\mathrm{supp}(f)$ and thus $j(x) \in /({}^*\mathrm{supp}(f), a) \subset {}^*S$. This proves the first equality.

Assume that $j(h) \circ j(x) \in {}^*\mathrm{supp}(f)$. Then $j(x) \in /({}^*\mathrm{supp}(f), j(h)) \approx /({}^*\mathrm{supp}(f), a) \subset {}^*U$. But $U \sqsubset S$ and $j(x) \in {}^*S$. So, we get the contradiction.

Since $a \circ j(x) \approx j(h) \circ j(x)$ if $j(x) \in \mathrm{ns}({}^*A)$ and $\mathrm{supp}(\varphi) \in j^{-1}({}^*S) \subset j^{-1}(\mathrm{ns}({}^*A))$ we have $\mathrm{supp}(\varphi) \approx 0$ and by the second inequality (7) $\Delta \sum\limits_{h \in H} \varphi(h) \approx 0$. This proves (8).

Let us show that the following inequality holds

$$^\circ\left(\Delta \sum_{x \in H} {}^*f(j(h \circ x)) - \Delta \sum_{x \in H} {}^*f(j(h) \circ j(x))\right) \geq 0. \tag{10}$$

By (9) we have

$$\Delta \sum_{x \in H} {}^*f(j(h \circ x)) - \Delta \sum_{x \in H} {}^*f(j(h) \circ j(x)) = \Delta \sum_{j(x) \notin {}^*S} {}^*f(j(h \circ x)) + \Delta \sum_{j(x) \in {}^*S} ({}^*f(j(h \circ x)) - {}^*f(j(h) \circ j(x)))$$

Obviously,

$$\Delta \sum_{j(x) \notin {}^*S} {}^*f(j(h \circ x)) = c \geq 0.$$

But

$$\Delta \sum_{j(x) \in {}^*S} ({}^*f(j(h \circ x)) - {}^*f(j(h) \circ j(x))) \approx 0.$$

Indeed, since $j(h), j(x) \in \mathrm{ns}({}^*A)$, when $j(x) \in {}^*S$ we have $j(h \circ x) \approx j(h) \circ j(h)$ by Theorem 5 and thus, ${}^*f(j(h \circ x)) \approx {}^*f(j(h) \circ j(h))$ by the continuity of f. Thus $\beta = \sup\limits_{j(x) \in {}^*S} |{}^*f(j(h \circ x)) - {}^*f(j(h) \circ j(x))| \approx 0$ and by Proposition 11

$$|\Delta \sum_{j(x) \in {}^*S} ({}^*f(j(h \circ x)) - {}^*f(j(h) \circ j(x)))| \leq C_{S,V} \beta \approx 0$$

Since $\{h \circ x \mid x \in H\}$ is a permutation of H we have

$$\Delta \sum_{x \in H} {}^*f(j(x)) = \Delta \sum_{x \in H} {}^*f(j(h \circ x))$$

Now

$$I(f) - I(l_a(f)) = {}^\circ\left(\Delta \sum_{x \in H} {}^*f(j(x)) - \Delta \sum_{x \in H} {}^*f(a \circ j(x))\right) =$$

$$= {}^\circ\left(\left(\Delta \sum_{x \in H} {}^*f(j(h \circ x)) - \Delta \sum_{x \in H} {}^*f(j(h) \circ j(x))\right) + \left(\Delta \sum_{x \in H} {}^*f(j(h) \circ j(x)) - \Delta \sum_{x \in H} {}^*f(a \circ j(x))\right)\right)$$

The first term on the right hand side of this equality is positive by (10), the second - infinitesimal by (8) and so $I(f) - I(l_a(f)) \geq 0$. $\square$

6 Proof of Theorem 3

First of all we will formulate some necessary results about the structure of finite semigroups from [14], where one can find the proofs.

Let S be a finite semigroup.

Definition 5 *1. $x \in S$ is said to be zero ($x = 0$) iff $\forall y \in S\ xy = yx = x$. (Obviously if the zero exists it is unique).*

2. *The set $I \subseteq S$ is a left (right) ideal iff $SI \subseteq I$ ($IS \subseteq I$). $I \subseteq S$ is an ideal iff I is a left and a right ideal. (Obviously an ideal (a left or a right ideal) is a subsemigroup.)*

3. *S is said to be 0-simple if it has no proper ideals but $\{0\}$ and $\emptyset$.*

4. *S is a zero semigroup iff $\forall s, t \in S\ st = 0$.*

5. *Let $I \subset S$ be an ideal of $(S, \cdot)$. The quotient semigroup S/I is the set $(S \backslash I) \cup \{0\}$ with multiplication "$*$" defined as the follows*
$$s_1 * s_2 = \begin{cases} s_1 \cdot s_2, & \textit{if}\ s_1 \cdot s_2 \notin I \\ 0, & \textit{if}\ s_1 \cdot s_2 \in I \end{cases}$$

6. *A maximal sequence of ideals for S is the ordered sequence of ideals of S*
$$S = I_0 \supset I_1 \supset I_2 ... I_n \supset I_{n+1} = \emptyset,$$
such that there are no ideals I' of S, $I_k \supset I' \supset I_{k+1}$.

It is clear that any finite semigroup has a maximal sequence of ideals.

Theorem 6 *Any semigroup I_{r-1}/I_r is 0-simple or zero.*

Let $n, m \in \mathbf{N}$, H be a group, $\rho : \{1, .., n\} \times \{1, .., m\} \to H \cup \{0\}$. Consider the Rees semigroup
$$S(n, m, H, \rho) = \{(i, j, h),\ \ i = 1, ..., n;\ j = 1, ..., m;\ h \in H\} \cup \{0\}$$
$$(i_1, j_1, h_1)(i_2, j_2, h_2) = \begin{cases} (i_1, j_2, h_1\rho(i_2, j_1)h_2), & \text{if } \rho(i_2, j_1) \in H \\ 0, & \text{if } \rho(i_2, j_1) = 0 \end{cases}$$

A Rees semigroup is called regular if $\forall i\ \exists\ j\ \rho(i, j) \neq 0$ and $\forall j \exists i\ \rho(i, j) \neq 0$.

Theorem 7 *Any finite 0-simple semigroup S (with zero) is isomorphic to a regular Rees semigroup.*

(If S is a semigroup without zero we may add zero to S or remove zero from the Rees semigroup.)
This theorem implies

Corollary 2 *Let S be a 0-simple finite semigroup, $0 \neq s \in S$ and $F = sSs$. Then F is a zero subsemigroup or, $F \backslash \{0\}$ is a group.*

Proof Let $s = (i_s, j_s, h_s)$. If $F \neq \{0\}$, then $F = \{sas,\ a \in S\} = \{(i_s, j_s, h),\ h \in H\} \cup \{0\}$. If $\rho(i_s, j_s) = 0$, then F is a zero semigroup; if $\rho(i_s, j_s) = g$, then $\phi : F \backslash \{0\} \to H$, $\phi(i_s, j_s, h) = hg$, is an isomorphism.

We are able now to prove Theorem 3.

Let G be a locally compact group that is approximable by finite semigroups and $< S, \phi >$ - a hyperfinite approximation of G by a hyperfinite semigroup S that exists by Theorem 5. We denote both operations - in G and in S by $\cdot$ since this does not lead to any misunderstanding.

Consider an internal hyperfinite maximal sequence of ideals in S (see definition 5(6))
$$S = I_0 \supset I_1 \supset I_2 ... I_n \supset I_{n+1} = \emptyset.$$
that exists by the transfer principle.

By assumptions, $\phi(S) \cap \mathrm{ns} \neq \emptyset$ and, consequently, there exists $r \in {}^*\mathbf{N}$ such that $\phi(I_{r-1}) \cap \mathrm{ns}({}^*G) \neq \emptyset$ and $\phi(I_r) \cap \mathrm{ns}({}^*G) = \emptyset$. There are two cases.

1. $I_r = \emptyset$. Then take $F = I_{r-1}$ and $\psi = \phi|_F$.

2. $I_r \neq \emptyset$. Then take $F = I_{r-1}/I_r$ and $\psi : F \to G$, $\psi(x) = \phi(x)$, for $x \neq 0$ and $\psi(0) = g \notin \mathrm{ns}({}^*G) \cup \mathrm{Im}(\phi)$.

Such a g exists by the following reasons. The group G is not compact since $\phi(I_r) \cap \mathrm{ns}({}^*G) = \emptyset$ and $I_r \neq \emptyset$, otherwise ${}^*G = \mathrm{ns}({}^*G)$ but $\phi(S) \subset G$. It is easy to see that there exist an internal compact $D \supset \mathrm{ns}({}^*G)$. The set ${}^*G \setminus {}^*D$ is not compact and thus not hyperfinite. So ${}^*G \setminus (\mathrm{ns}({}^*G) \cup \mathrm{Im}(\phi)) \neq \emptyset$.

Let us prove only that $< F, \psi >$ approximates G in the sense of theorem 5. Let us denote by $*$ the operation on F.

First, we will show that ψ is an almost homomorphism, that is
$\forall x, y \in F\ (\psi(x), \psi(y) \in \mathrm{ns}({}^*G)) \Longrightarrow \psi(x * y) \approx \psi(x)\psi(y)$. Let $x, y \in F$ and $\psi(x), \psi(y) \in \mathrm{ns}({}^*G)$, we have to prove that $\psi(x*y) \approx \psi(x)\psi(y)$. For the case 1) it is trivial, since ψ is a restriction of ϕ on subsemigroup. Consider case 2). Since $\psi(x), \psi(y) \in \mathrm{ns}({}^*G)$, one has $x, y \neq 0$, so, $\psi(x) = \phi(x)$ and $\psi(y) = \phi(y)$. Then $\mathrm{ns}({}^*G) \ni \phi(x)\phi(y) \approx \phi(xy)$. So, $\phi(xy) \in \mathrm{ns}$ and thus $xy \notin I_r$. By the definition of the operation in a quotient semigroup $x * y = xy \neq 0$, and by the construction $\psi(x * y) = \phi(xy)$.

It remains to prove that $\forall g \in G\ \exists x \in F\ g \approx \psi(x)$ or, the same, $\forall g \in G \exists x \in I_{r-1}\ g \approx \phi(x)$. Since $\phi(I_{r-1}) \cap \mathrm{ns}({}^*G) \neq \emptyset$, there exists an $x \in I_{r-1}$ such that $\phi(x) \in \mathrm{ns}({}^*G)$. Since ${}^{-1}$ is a continuous operation $(\phi(x))^{-1} \in \mathrm{ns}({}^*G)$ and there exists $y \in S\ \phi(y) \approx (\phi(x))^{-1}$. So, $e \approx \phi(y)\phi(x) \approx \phi(yx)$. Notice that $yx \in I_{r-1}$ since I_{r-1} is an ideal and $x \in I_{r-1}$. Now, let $g \in G$ and $s \in S$ such that $\phi(s) \approx g$. Then $\phi(yxs) \approx g$ and $yxs \in I_{r-1}$.

Obviously a zero semigroup can never approximate an infinite group, and thus F is 0-simple hyperfinite semigroup by Theorem 6

Let $s \in F$ be such that $\psi(s) \approx e$. Consider the semigroup $T = sFs$. It is easy to see if $j = \psi|_T$ then the pair $< T, j >$ approximates G.

By Corollary 2 of Theorem 7 $H = T \setminus \{0\}$, is a hyperfinite group. If T does not contain 0, then the proof is done. Suppose $0 \in T$. Then it is enough to prove that $j(0) \notin \mathrm{ns}({}^*G)$.

Suppose that $j(0) \in \mathrm{ns}({}^*G)$. If $j(0) \approx e$, then $\forall x \in G\ xe \approx e$ which is impossible. If $j(0) \approx x$ and $x \neq e$, then there exist y such that $j(y) \approx x^{-1}$. Now
$e = xx^{-1} \approx j(0)j(y) \approx j(0 * y) = j(0) \approx x$. This is impossible since x, e are standard and $x \neq e$. $\square$

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1991 *Mathematics Subject Classification.* Primary 26E35, 03H05; Secondary 28E05, 42A38